On the Spectra of P- and P_0-Matrices

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ABSTRACT

We consider some conjectures of D. Hershkowitz and A. Berman in concerning the localization of eigenvalues of P- and P_0-matrices. We show that two of these conjectures are false. We improve a classical formula of Routh's to prove our results. Also we reconsider a theorem of Kellogg's with the generalized result.

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1. INTRODUCTION

DEFINITION 1. A P-matrix [P_0-matrix] is an $n\times n$ complex matrix all of whose principal minors are positive [non-negative].

Such matrices are related to stable matrices and play an important role in economics and mathematical programming.

Let $S=\{u_1,\dots,u_n\}$, where $u_1,\dots,u_n\in C$ (the complex field), and let

$$\sigma_k(S)=\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n}\prod_{t=1}^{k}u_{i_t},\qquad k=1,\dots,n,$$

denote the kth elementary symmetric function of the numbers $u_1,\dots,u_n$. The set S is called a P-set [P_0-set] if it satisfies

$$\sigma_k(S)>0,\qquad k=1,\dots,n$$

$$\left[\sigma_k(S)\geqslant 0,\qquad k=1,\dots,n\right].$$

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It is shown in [3] that S is a P-set [P_0-set] if and only if it is the spectrum of some P-matrix [P_0-matrix]. In order to investigate the localization of the spectrum of a P-matrix [P_0-matrix], some authors have utilized P-sets [P_0-sets], e.g. in [6], [4], [3], and [5].

Observe also that S is a P_0-set if and only if $S \cup \{0\}$ is a P_0-set. Thus we consider in this paper P_0-sets which do not contain zero elements.

Kellogg [6] proved that elements of a P-set [P_0-set] cannot lie in a certain wedge around the negative axis. More precisely, he proved:

THEOREM A (Kellogg [6]).

(I) *If* $\{u_1, \ldots, u_n\}$ *is a P-set, then*

$$|\arg u_i| < \pi - \frac{\pi}{n}, \qquad i = 1, \ldots, n. \tag{1}$$

(II) *If* $S = \{u_1, \ldots, u_n\}$, $u_i \neq 0$, $i = 1, \ldots, n$, *is a* P_0-*set, then*

$$|\arg u_i| \leqslant \pi - \frac{\pi}{n}, \qquad i = 1, \ldots, n. \tag{1'}$$

Equality in (1′) *holds iff*

$$\sigma_k(S) = 0, \quad k = 1, \ldots, n-1, \qquad \sigma_n(S) > 0.$$

For a set S, we denote by $\pi(S)$ *and* $\surd(S)$ *the numbers of elements of S which have positive and negative real parts respectively.*

A natural question is whether for a P-set or a P_0-*set* $S = \{u_1, \ldots, u_n\}$ *such that* $\pi(S)$ *or* $\surd(S)$ *is given, the region described by* (1) *can be reduced, i.e. whether there exists a number* α *satisfying*

$$|\arg u_i| < \alpha < \pi - \frac{\pi}{n}, \qquad i = 1, \ldots, n.$$

In [4] *the following theorems are shown*:

THEOREM B. *Let* $S = \{u_1, \ldots, u_n\}$ *be a P-set such that* $\pi(S) = 1$. *Then*

$$|\arg u_i| < \tfrac{2}{3}\pi, \qquad i = 1, \ldots, n.$$

THEOREM C. *Let $S=\{u_1,\dots,u_n\}$, $u_i\neq 0$, $i=1,\dots,n$, be a P_0-set such that $\pi(S)=1$. Then*

$$|\arg u_i| \leqslant \tfrac{2}{3}\pi,\ i=1,\dots,n. \tag{2}$$

Equality in (2) holds for $i=j$ only if

$$S=\{\,|u_j|, u_j, \bar{u}_j\}\cup S_1,$$

where S_1 consists of pairs of conjugate pure imaginary numbers.

As generalizations of Theorem B and C, D. Hershkowitz and A. Berman [4] conjecture

CONJECTURE 1. Let $S=\{u_1,\dots,u_n\}$ be a P-set such that $\pi(S)=2$. Then

$$|\arg u_i| < \tfrac{5}{6}\pi, \qquad i=1,\dots,n.$$

CONJECTURE 2. Let $S=\{u_1,\dots,u_n\}$ be a P-set such that $\nu(S)=2$. Then

$$|\arg u_i| < \tfrac{5}{6}\pi, \qquad i=1,\dots,n.$$

More generally, they conjecture

CONJECTURE 3. Let $S=\{u_1,\dots,u_n\}$ be a P-set such that $\pi(S)=k$. Then

$$|\arg u_i| < \frac{2k}{2k+1}\pi, \quad i=1,\dots,n, \qquad \text{if } k \text{ is odd};$$

$$|\arg u_i| < \frac{2k+1}{2k+2}\pi, \quad i=1,\dots,n, \qquad \text{if } k \text{ is even}.$$

CONJECTURE 4. Let $S=\{u_1,\dots,u_n\}$ be a P-set such that $\nu(S)=k$ (k must be even). Then

$$|\arg u_i| < \frac{2k+1}{2k+2}\pi, \qquad i=1,\dots,n.$$

Similar conjectures may be stated for P_0-sets by replacing "$<$" with "$\leqslant$." These conjectures will be discussed in this paper. For this purpose, in Section 2, we generalize Routh's classical formula in [2, p. 178]. We obtain a formula for the number of roots which lie in the fan region determined by $\{z : |\arg z| < \theta\}$, $0 < \theta < \pi$, for a polynomial with real coefficients. Moreover, in Section 3, making use of this formula, we show that Conjectures 1 and 2 are false. Meantime, in Section 2, a new proof of Theorem A (Kellogg) is given.

2. THE LOCATIONS OF THE ROOTS OF POLYNOMIALS WITH REAL COEFFICIENTS

DEFINITION 2. The Cauchy index of a real rational function $R(x)$ between the limits a and b [notation: $I_a^b R(x)$; a and b are real numbers or $\pm\infty$] is the difference between the number of jumps of $R(x)$ from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ as the argument changes from a to b.

REMARK. In counting the number of jumps, the extreme values of x (that is, the limits a and b) are not included.

Let $x_1, \dots, x_s$ be all jumps of $R(x)$ between the limits a and b. We introduce the notation

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ is a jump of } R(x) \text{ from } -\infty \text{ to } +\infty; \\ -1 & \text{if } x \text{ is a jump of } R(x) \text{ from } +\infty \text{ to } -\infty. \end{cases}$$

Then it is easy to see that

$$I_a^b R(x) = \sum_{i=1}^{s} \delta(x_i).$$

Let $f(Z) = a_0 Z^n + a_1 Z^{n-1} + \cdots + a_n$ $(a_0 \neq 0)$ be a polynomial with real coefficients, and let θ satisfy $0 < \theta < \pi$. Let K be the number of roots u of $f(Z)$ such that $|\arg u| < \theta$.

If $\theta = \pi/2$, then under the assumption that $f(Z)$ has no roots on the imaginary axis, the formula for the number K of roots u of $f(Z)$ such that

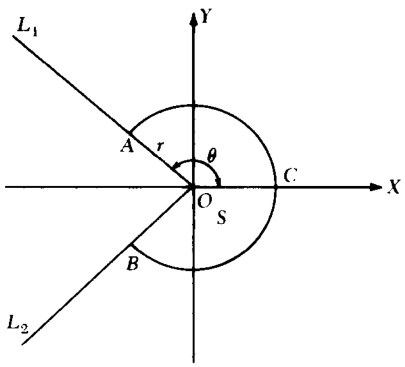

FIG. 1.

$|\arg u| < \pi/2$, that is, the number of roots of $f(Z)$ in the right half plane, is (see [2, p. 178])

$$I_{-\infty}^{+\infty} \frac{a_1 w^{n-1} - a_3 w^{n-3} + \cdots}{a_0 w^n - a_2 w^{n-2} + \cdots} = n - 2K, \qquad -\infty < w < +\infty.$$

We shall give the formula for the number K of roots u of $f(Z)$ for any θ such that $|\arg u| < \theta$.

In the complex plane we construct the circle of radius r with its center at the origin. We have two rays $L_1: y = x \tan\theta$ $(y \geqslant 0)$ and $L_2: y = -x\tan\theta$ $(y \leqslant 0)$. Assume that the circle intersects L_1, L_2, and the right real semiaxis at A, B, and C respectively. Let S denote the domain bounded by the segments OA, OB and the circular arc $\widehat{BCA}$ (see Figure 1). Then when r is large enough, all roots u of $f(Z)$ such that $|\arg u| < \theta$ lie in the domain S.

We begin with the case where $f(Z)$ has no roots on the rays L_1 and L_2. By the argument principle in classical complex analysis (see [1]), $\arg f(Z)$ increases by

$$\Delta_{\Gamma_{S(r)}} \arg f(Z) = 2K\pi \tag{3}$$

if the point Z moves in the counterclockwise direction around the contour, denoted by $\Gamma_{S(r)}$, of the domain S (see Figure 1).

Let T_r denote the circular arc $\widehat{BCA}$. It is easy to see that

$$\lim_{r\to\infty} \Delta_{T_r} \arg f(Z) = 2n\theta. \tag{4}$$

Let $\Delta_{BOA(r)} \arg f(Z)$ be the increment of the argument of $f(Z)$ from B to 0 around L_2, then to A around L_1. Similarly, $\Delta_{OA(r)} \arg f(Z)$, $\Delta_{BO(r)} \arg f(Z)$, and so on are defined. Then

$$\Delta_{\Gamma_{S(r)}} \arg f(Z) = \Delta_{T_r} \arg f(Z) - \Delta_{BOA(r)} \arg f(Z). \tag{5}$$

Let $r \to \infty$ in (5); then it follows from (3) and (4) that

$$2K\pi = 2n\theta - \Delta_{BOA(\infty)} \arg f(Z),$$

where

$$\Delta_{BOA(\infty)} \arg f(Z) = \lim_{r \to \infty} \Delta_{BOA(r)} \arg f(Z).$$

Hence

$$\Delta_{BOA(\infty)} \arg f(Z) = -2K\pi + 2n\theta. \tag{6}$$

Let Z_A and Z_B be the coordinates of the points A and B respectively. It is easy to prove that $Z_A = \overline{Z}_B$, then $f(Z_A) = \overline{f(Z_B)}$. When r moves so that A, B turn out to be A', B' respectively, then $f(Z_{A'}) = \overline{f(Z_{B'})}$. Then

$$\Delta_A^{A'} \arg f(Z) = -\Delta_B^{B'} \arg f(Z)$$

(see Figure 2). Furthermore

$$\Delta_{OA(r)} \arg f(Z) = -\Delta_{OB(r)} \arg f(Z). \tag{7}$$

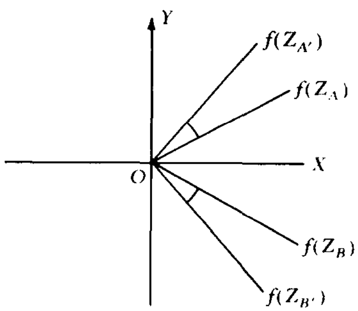

FIG. 2.

Also

$$\begin{aligned}\Delta_{BOA(r)} \arg f(Z) &= \Delta_{BO(r)} \arg f(Z) + \Delta_{OA(r)} \arg f(Z)\\ &= -\Delta_{OB(r)} \arg f(Z) + \Delta_{OA(r)} \arg f(Z)\\ &= 2\Delta_{OA(r)} \arg f(Z) \qquad [\text{from } (7)].\end{aligned}$$

Then

$$\begin{aligned}\Delta_{OA(\infty)} \arg f(Z) &= \lim_{r\to\infty} \Delta_{OA(r)} \arg f(Z) = \lim_{r\to\infty} \tfrac{1}{2}\Delta_{BOA(r)} \arg f(Z)\\ &= n\theta - K\pi \qquad [\text{from } (6)]; \end{aligned} \tag{8}$$

substituting $Z_A = re^{\theta i}$ for the variable Z of $f(Z)$, we get

$$\begin{aligned}F(r) = f(Z_A) &= a_0 r^n e^{n\theta i} + a_1 r^{n-1} e^{(n-1)\theta i} + \cdots + a_{n-1} r e^{\theta i} + a_n\\ &= \left(a_0 r^n \cos n\theta + a_1 r^{n-1} \cos(n-1)\theta + \cdots + a_{n-1} r\cos\theta + a_n\right)\\ &\quad + i\left(a_0 r^n \sin n\theta + a_1 r^{n-1}\sin(n-1)\theta + \cdots + a_{n-1} r \sin\theta\right).\end{aligned}$$

Let

$$\begin{aligned}U(r) &= a_0 r^n \cos n\theta + \cdots + a_{n-1} r\cos\theta + a_n,\\ W(r) &= a_0 r^n \sin n\theta + \cdots + a_{n-1} r\sin\theta;\end{aligned}$$

then

$$F(r) = U(r) + iW(r). \tag{9}$$

Because no root of $f(Z)$ lies on L_1 and L_2, $F(0) = f(0) = a_n \neq 0$ and

$$\arg F(0) = \begin{cases} 0 & \text{if} \quad a_n > 0,\\ \pi & \text{if} \quad a_n < 0.\end{cases} \tag{10}$$

Let $r_1, \ldots, r_s$ be all the jumps of $W(r)/U(r)$ between the limits 0 and $+\infty$; then

$$\Delta_0^{\infty} \arg F(r) = \Delta_0^{r_1} \arg F(r) + \Delta_{r_1}^{r_2} \arg F(r) + \Delta_{r_s}^{+\infty} \arg F(r).$$

It is known that

$$\Delta_0^{+\infty} \arg F(r) = \Delta_{OA(\infty)} \arg f(Z).$$

By (8),

$$\Delta_0^{\infty} \arg F(r) = n\theta - K\pi. \tag{11}$$

Since $r_1, \dots, r_s$ are jumps of $W(r)/U(r)$, we have $U(r_i) = 0$, $i = 1, \dots, s$. And since no root of $f(Z)$ lies on L_1 and L_2, $F(r_i) \neq 0$. Hence $W(r_i) \neq 0$, that is, all $F(r_i)$ $(\neq 0)$ lie on the imaginary axis, $i = 1, \dots, s$.

Now, we will consider the relation between $\Delta_0^{\infty} \arg F(r)$ and $I_0^{\infty} W(r)/U(r)$. Notice that in this paper $\arg F(r)$ denotes the principal value of the argument of $F(r)$ and $-\pi < \arg F(r) \leqslant \pi$.

LEMMA 1. $\Delta_{r_1}^{r_s} \arg F(r) = -\pi I_{r_1}^{r_s} W(r)/U(r) - \frac{1}{2}[\delta(r_1) + \delta(r_s)]\pi.$

Proof. We use induction on s.

If $s = 2$, no jump is between the limits r_1 and r_2. Hence

$$I_{r_1}^{r_2} \frac{W(r)}{U(r)} = 0. \tag{12}$$

$F(r_i)$ $(\neq 0)$, $i = 1, 2$, is on the imaginary axis for jumps r_i. Let $\delta(r_i) = 1$; then when r moves from left to right through r_i, $F(r)$ shifts from quadrant II to quadrant I through the upper imaginary semiaxis, or from quadrant IV to quadrant III through the lower imaginary semiaxis. Let $\delta(r_i) = -1$; then when r moves from left to right through r_i, $F(r)$ shifts from quadrant I to quadrant II through the upper imaginary semiaxis, or from quadrant III to quadrant IV through the lower imaginary semiaxis. Hence we obtain

(i) If $\delta(r_1) = \delta(r_2) = 1$, then $\Delta_{r_1}^{r_2} \arg F(r) = -\pi = -\pi I_{r_1}^{r_2} W(r)/U(r) - \frac{1}{2}(1+1)\pi$;

(ii) If $\delta(r_1) = -1$, $\delta(r_2) = 1$, then $\Delta_{r_1}^{r_2} \arg F(r) = 0 = -\pi I_{r_1}^{r_2} W(r)/U(r) - \frac{1}{2}(-1+1)\pi$;

(iii) If $\delta(r_2) = -1$, $\delta(r_1) = 1$, then $\Delta_{r_1}^{r_2} \arg F(r) = 0 = -\pi I_{r_1}^{r_2} W(r)/U(r) - \frac{1}{2}[1+(-1)]\pi$;

(iv) If $\delta(r_1) = \delta(r_2) = -1$, then $\Delta_{r_1}^{r_2} \arg F(r) = \pi = -\pi I_{r_1}^{r_2} W(r)/U(r) - \frac{1}{2}[(-1)+(-1)]\pi$.

Combining the four cases, we have

$$\Delta_{r_1}^{r_2} \arg F(r) = -\pi I_{r_1}^{r_2} \frac{W(r)}{U(r)} - \tfrac{1}{2}[\delta(r_1) + \delta(r_2)]\pi. \tag{13}$$

Therefore, if $s = 2$, the lemma holds.

Suppose that $s > 2$, and that the lemma holds for $s = m - 1$. Then

$$\Delta_{r_1}^{r_{m-1}} \arg F(r) = -\pi I_{r_1}^{r_{m-1}} \frac{W(r)}{U(r)} - \tfrac{1}{2}[\delta(r_1) + \delta(r_{m-1})]\pi. \tag{14}$$

For $s = m$,

$$\begin{aligned}
\Delta_{r_1}^{r_m} \arg F(r) &= \Delta_{r_1}^{r_{m-1}} \arg F(r) + \Delta_{r_{m-1}}^{r_m} \arg F(r) \\
&= -\pi I_{r_1}^{r_{m-1}} \frac{W(r)}{U(r)} - \tfrac{1}{2}[\delta(r_1) + \delta(r_{m-1})]\pi \\
&\quad -\pi I_{r_{m-1}}^{r_m} \frac{W(r)}{U(r)} - \tfrac{1}{2}[\delta(r_{m-1}) + \delta(r_m)]\pi \\
&\qquad\qquad \text{(from (14) and (13))} \\
&= -\left(I_{r_1}^{r_m} \frac{W(r)}{U(r)} - \delta(r_{m-1}) \right)\pi - \tfrac{1}{2}[\delta(r_1) + \delta(r_m)]\pi - \delta(r_{m-1})\pi \\
&\qquad \left(\text{because} \quad I_{r_1}^{r_m} \frac{W(r)}{U(r)} = I_{r_1}^{r_{m-1}} \frac{W(r)}{U(r)} + \delta(r_{m-1}) \right) \\
&= -I_{r_1}^{r_m} \frac{W(r)}{U(r)} - \tfrac{1}{2}[\delta(r_1) + \delta(r_m)]\pi.
\end{aligned}$$

Therefore, the lemma holds. ■

LEMMA 2.

(i) *If* $\arg F(r_s) = (\pi/2)\delta(r_s)$, *then*

$$-\frac{\pi}{2} \leqslant \lim_{r\to\infty} \arg F(r) \leqslant \frac{\pi}{2} \quad \text{and} \quad \Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arg F(r) - \frac{\pi}{2}\delta(r_s).$$

(ii) *If* $\arg F(r_s) = -(\pi/2)\,\delta(r_s)$, *then*

$$-\frac{\pi}{2} \geqslant \lim_{r\to\infty} \arg F(r) \geqslant -\pi \quad \textit{or} \quad \pi \geqslant \lim_{r\to\infty} \arg F(r) \geqslant \frac{\pi}{2}.$$

When $\lim_{r\to\infty} \arg F(r)$ *is between* $-(\pi/2)\,\delta(r_s)$ *and* $-\pi\,\delta(r_s)$, *then*

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arg F(r) + \frac{\pi}{2}\,\delta(r_s);$$

when $\lim_{r\to\infty} \arg F(r)$ *is between* $(\pi/2)\,\delta(r_s)$ *and* $\pi\,\delta(r_s)$, *then*

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arg F(r) - \tfrac{3}{2}\pi\,\delta(r_s).$$

Proof. First, assume that $\delta(r_s) = 1$. It is known that $F(r_s)$ $(\neq 0)$ is on the imaginary axis. No jump of $W(r)/U(r)$ lies in $(r_s, +\infty)$.

(i): If $F(r_s)$ lies on the upper imaginary semiaxis, that is, $\arg F(r_s) = \pi/2$, then for any $r \in (r_s, +\infty)$, $F(r)$ lies in quadrant I or IV, that is,

$$-\frac{\pi}{2} < \arg F(r) < \frac{\pi}{2} \qquad \text{for} \quad r \in (r_s, +\infty).$$

Therefore

$$-\frac{\pi}{2} \leqslant \lim_{r\to\infty} \arg F(r) \leqslant \frac{\pi}{2}$$

and

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arg F(r) - \arg F(r_s) = \lim_{r\to\infty} \arg F(r) - \frac{\pi}{2}.$$

(ii): If $F(r_s)$ lies on the lower imaginary semiaxis, that is, $\arg F(r_s) = -\pi/2$, then for $r \in (r_s, +\infty)$, $F(r)$ lies in quadrant II or III, that is,

$$\frac{\pi}{2} < \arg F(r) \leqslant \pi \quad \text{or} \quad -\pi < \arg F(r) < -\frac{\pi}{2}.$$

Therefore

$$\frac{\pi}{2} \leqslant \lim_{r \to \infty} \arg F(r) \leqslant \pi \quad \text{or} \quad -\pi \leqslant \lim_{r \to \infty} \arg F(r) \leqslant -\frac{\pi}{2}.$$

When $-\pi \leqslant \lim_{r \to \infty} \arg F(r) \leqslant -\pi/2$, it follows from $\arg F(r_s) = -\pi/2$ that

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r \to \infty} \arg F(r) - \left(-\frac{\pi}{2}\right) = \lim_{r \to \infty} \arg F(r) + \frac{\pi}{2}.$$

When $\pi/2 \leqslant \lim_{r \to \infty} \arg F(r) \leqslant \pi$, then

$$\begin{aligned}\Delta_{r_s}^{\infty} \arg F(r) &= \left(\lim_{r \to \infty} \arg F(r) - 2\pi\right) - \arg F(r_s) \\ &= \lim_{r \to \infty} \arg F(r) - \tfrac{3}{2}\pi.\end{aligned}$$

Therefore, Lemma 2 holds if $\delta(r_s) = 1$.
Similarly, we can prove Lemma 2 for $\delta(r_s) = -1$. ∎

LEMMA 3.

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r \to \infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2}\,\delta(r_s).$$

Proof.

(1) If $\arg F(r_s) = (\pi/2)\,\delta(r_s)$, then by Lemma 2,

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r \to \infty} \arg F(r) - \frac{\pi}{2}\,\delta(r_s).$$

But since $-\pi/2 < \arg F(r) < \pi/2$, we have that

$$argF(r) = arctan\,[W(r)/U(r)],$$

where $r \in (r_s, +\infty)$. Hence

$$\lim_{r\to\infty} \arg F(r) = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)}.$$

Therefore

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2}\,\delta(r_s).$$

(2) If $\arg F(r_s) = -(\pi/2)\,\delta(r_s)$, then by Lemma 2,

$$-\frac{\pi}{2} > \arg F(r) > -\pi \quad \text{or} \quad \frac{\pi}{2} < \arg F(r) < \pi, \qquad \text{where} \quad r \in (r_s, +\infty).$$

Hence

$$arg F(r) - arctan\left[W(r)/U(r)\right] = -\pi \quad or$$

$$arg F(r) - arctan\left[W(r)/U(r)\right] = \pi$$

for $r \in (r_s, +\infty)$, and

$$\lim_{r\to\infty} \arg F(r) - \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} = -\pi \quad \text{or}$$

$$\lim_{r\to\infty} \arg F(r) - \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} = \pi. \tag{15}$$

Now let $\delta(r_s) = 1$.

(i) When $-(\pi/2) \geqslant \lim_{r\to\infty} \arg F(r) \geqslant -\pi$, by Lemma 2,

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arg F(r) + \frac{\pi}{2}.$$

And by (15),

$$\lim_{r\to\infty} \arg F(r) = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \pi.$$

Then

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2} = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2}\delta(r_s).$$

(ii) When $\pi/2 \leqslant \lim_{r\to\infty} \arg F(r) \leqslant \pi$, by Lemma 2,

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arg F(r) - \tfrac{3}{2}\pi.$$

By (15),

$$\lim_{r\to\infty} \arg F(r) = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} + \pi.$$

Then

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2} = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2}\delta(r_s).$$

Similarly, let $\delta(r_s) = -1$; then

$$\Delta_{r_s}^{\infty} \arg F(r) = \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2}\delta(r_s).$$

Therefore, Lemma 3 holds. ■

We have that $\Delta_0^{\infty} \arg F(r) = \Delta_0^{r_1} \arg F(r) + \Delta_{r_1}^{r_s} \arg F(r) + \Delta_{r_s}^{\infty} \arg F(r)$, where $\Delta_0^{r_1} \arg F(r) = -(\pi/2)\delta(r_1)$. By Lemmas 1 and 3,

$$\begin{aligned}
\Delta_0^{\infty} \arg F(r) &= -\frac{\pi}{2}\delta(r_1) - \pi I_{r_1}^{r_s}\frac{W(r)}{U(r)} - \tfrac{1}{2}[\delta(r_1) + \delta(r_s)]\pi \\
&\quad + \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - \frac{\pi}{2}\delta(r_s) \\
&= -I_{r_1}^{r_s}\frac{W(r)}{U(r)} + \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)} - [\delta(r_1) + \delta(r_s)]\pi.
\end{aligned}$$

And because $I_0^\infty W(r)/U(r) = I_{r_1}^{r_s} W(r)/U(r) + \delta(r_1) + \delta(r_s)$, we obtain that

$$\Delta_0^\infty \arg F(r) = -\pi I_0^\infty \frac{W(r)}{U(r)} + \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)}.$$

Since $\Delta_0^\infty \arg F(r) = n\theta - K\pi$ by (11), then

$$n\theta - K\pi = -\pi I_0^\infty \frac{W(r)}{U(r)} + \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)}.$$

Moreover,

$$K = I_0^\infty \frac{W(r)}{U(r)} + \frac{n\theta}{\pi} - \frac{1}{\pi} \lim_{r\to\infty} \arctan \frac{W(r)}{U(r)}. \tag{16}$$

By [2, Section 15.2], $I_0^\infty W(r)/U(r) = V(0) - V(\infty)$, where $V(0)$ is the number of variations of sign in the Sturm chain constructed by $W(r)$ and $U(r)$ at $r = 0$, and $V(\infty)$ is the number of variations of sign for large enough r.

From (16) and

$$lim_{r\to\infty}\, arctan[W(r)/U(r)] = arctan \lim_{r\to\infty} [W(r)/U(r)],$$

we obtain the following

THEOREM 1. *Let $f(Z) = a_0 Z^n + a_1 Z^{n-1} + \cdots + a_n$ $(a_0 \neq 0)$ be a real polynomial. Assume that no root of $f(Z)$ lies on $L_1: y = x\tan\theta$ $(y \geqslant 0)$ or $L_2: y = -x\tan\theta$ $(y \leqslant 0)$, $0 < \theta < \pi$. Then the number K of roots u of $f(Z)$ such that $|\arg u| < \theta$ is*

$$K = I_0^\infty \frac{W(r)}{U(r)} + \frac{n\theta}{\pi} - \frac{1}{\pi} \arctan \lim_{r\to\infty} \frac{W(r)}{U(r)},$$

where

$$W(r) = a_0 r^n \sin n\theta + a_1 r^{n-1} \sin(n-1)\theta + \cdots + a_{n-1} r \sin\theta,$$

$$U(r) = a_0 r^n \cos n\theta + a_1 r^{n-1} \cos(n-1)\theta + \cdots + a_{n-1} r\cos\theta + a_n.$$

Equivalently,

$$K = V(0) - V(\infty) + \frac{n\theta}{\pi} - \frac{1}{\pi}\arctan \lim_{r\to\infty} \frac{W(r)}{U(r)}. \qquad \blacksquare$$

The Sturm chain constructed by $W(r)$ and $U(r)$ can be determined by the Euclidean algorithm (see [2, Section 15.2]).

If $\cos n\theta \neq 0$, then $\arctan \lim_{r\to\infty}[W(r)/U(r)] = \arctan\tan n\theta$. Hence we obtain

Corollary 1. *Under the conditions of Theorem* 1, *if* $\cos n\theta \neq 0$, *then*

$$K = I_0^\infty \frac{W(r)}{U(r)} + \frac{n\theta}{\pi} - \frac{1}{\pi}\arctan\tan n\theta.$$

Theorem 1 is deduced under the assumption that $f(Z)$ has no roots on the rays L_1 and L_2. Now we shall consider the case where the polynomial $f(Z) = a_0Z^n + \cdots + a_n$ $(a_0 \neq 0)$ has s roots on the rays L_1 and L_2.

If Z_0 is a root on L_1 of $f(Z)$, then $f(Z_0) = 0$ implies $f(\bar{Z}_0) = 0$, that is, $\bar{Z}_0$ on L_2 is also a root of $f(Z)$. Hence the roots on L_1 or L_2 of $f(Z)$ consist of some pairs of conjugate roots of $f(Z)$. Let them be $Z_1, \ldots, Z_s$. Then

$$d(Z) = (Z - Z_1)\cdots(Z - Z_s) = Z^s + \cdots$$

is a polynomial with real coefficients. There exists $f^*(Z)$ such that $f(Z) = d(Z)f^*(Z)$. None of the $n^* = n - s$ roots of $f^*(Z)$ is on L_1 or L_2. The roots of $f(Z)$ consist of conjugate pairs. So do the roots of $f^*(Z)$. Hence $(1/a_0)f^*(Z) = Z^{n-s} + \cdots$ is a polynomial with real coefficients. So is $f^*(Z) = a_0(1/a_0)f^*(Z)$.

Let $F^*(r) = f^*(re^{i\theta}) = U^*(r) + iW^*(r)$, where θ is fixed, and $U^*(r)$ and $W^*(r)$ are two polynomials with real coefficients. By Theorem 1,

$$K = I_0^\infty \frac{W^*(r)}{U^*(r)} + \frac{(n-s)\theta}{\pi} - \frac{1}{\pi}\arctan \lim_{r\to\infty} \frac{W^*(r)}{U^*(r)}.$$

Therefore, we have, for $0 < \theta < \pi$, the following

Theorem 2. *Let* $f(Z) = a_0Z^n + a_1Z^{n-1} + \cdots + a_n$ $(a_0 \neq 0)$ *be a polynomial with real coefficients. Assuming that the number of roots on* L_1 *or* L_2 *of* $f(Z)$ *is just* s, *where* $L_1\colon y = x\tan\theta$ $(y \geqslant 0)$ *and* $L_2\colon y = -x\tan\theta$ $(y \leqslant 0)$, $0 < \theta < \pi$. *Then there exist real polynomials* $d(Z)$ *and* $f^*(Z)$ *such that* $f(Z) = d(Z)f^*(Z)$, *and* $f^*(Z)$ *with the leading coefficients* a_0 *has no* *roots on L_1 or L_2. Let $f^*(re^{i\theta}) = U^*(r) + iW^*(r)$, where $U^*(r)$ and $W^*(r)$ are two polynomials with real coefficients. Then the number K of roots u of $f(Z)$ such that $|\arg u| < \theta$ is*

$$K = I_0^{\infty}\frac{W^*(r)}{U^*(r)} + \frac{(n-s)\theta}{\pi} - \frac{1}{\pi}\arctan \lim_{r\to\infty}\frac{W^*(r)}{U^*(r)}.$$

COROLLARY 2. *Under the conditions of Theorem 2, if $\cos n\theta \neq 0$, then*

$$K = I_0^{\infty}\frac{W^*(r)}{U^*(r)} + \frac{(n-s)\theta}{\pi} - \frac{1}{\pi}\arctan\tan[(n-s)\theta].$$

Note that since

$$\begin{aligned} f(re^{i\theta}) &= U(r) + iW(r) = d(re^{i\theta})f^*(re^{i\theta}) \\ &= d(re^{i\theta})[U^*(r) + iW^*(r)], \end{aligned}$$

in general, $d(re^{i\theta})$ with the variable r is not a polynomial with real coefficients. Hence $W^*(r)/U^*(r)$ and $W(r)/U(r)$ may not be identical. Moreover, $I_0^{\infty}W^*(r)/U^*(r)$ and $I_0^{\infty}W(r)/U(r)$ may not be identical. This is different from the situation for $\theta = \pi/2$ discussed in [2].

As an application of the results we have just obtained, we give a new proof of Theorem A.

Proof of Theorem A.

(I) $S = \{u_1, \dots, u_n\}$ is the set of all roots of

$$f(Z) = Z^n - \sigma_1 Z^{n-1} + \cdots + (-1)^{n-1}\sigma_{n-1}Z + (-1)^n\sigma_n$$

where $\sigma_k = \sigma_k(S) > 0$, $k = 1, \dots, n$. Let $\theta = \pi - \pi/n$ in Theorem 1; then

$$\begin{aligned} W(r) = {}& r^n \sin n\left(\pi - \frac{\pi}{n}\right) - \sigma_1 r^{n-1}\sin(n-1)\left(\pi - \frac{\pi}{n}\right) + \cdots \\ & + (-1)^{n-1}\sigma_{n-1} r \sin\left(\pi - \frac{\pi}{n}\right), \end{aligned}$$

$$\begin{aligned} U(r) = {}& r^n \cos n\left(\pi - \frac{\pi}{n}\right) - \sigma_1 r^{n-1}\cos(n-1)\left(\pi - \frac{\pi}{n}\right) + \cdots \\ & + (-1)^{n-1}\sigma_{n-1} r \cos\left(\pi - \frac{\pi}{n}\right) + (-1)^n\sigma_n. \end{aligned}$$

(i) If n is even, then the coefficient of r^{n-k} in $W(r)$ (let $\sigma_0 = 1$) is

$$(-1)^k \sigma_k \sin(n-k)\left(\pi - \frac{\pi}{n}\right)$$

$$= (-1)^k \sigma_k \sin\left((n-k)\pi - \frac{n-k}{n}\pi\right)$$

$$= \begin{cases} (-1)^k \sigma_k \sin \dfrac{n-k}{n}\pi & \text{if} \quad k = 1,3,\dots,n-1, \\ (-1)^{k+1} \sigma_k \sin \dfrac{n-k}{n}\pi & \text{if} \quad k = 0,2,\dots,n-2 \end{cases}$$

$$= \; -\sigma_k \sin \frac{n-k}{n}\pi \qquad \text{for} \quad k = 0,1,\dots,n-1$$

$$\begin{cases} = 0 & \text{if} \quad k = n, \\ < 0 & \text{if} \quad k = 1,\dots,n-1. \end{cases}$$

Hence for any $r \in (0, +\infty)$, $W(r) < 0$. The coefficients of $r^n, r^{n-1}, \dots, r, 1$ in $U(r)$ are respectively

$$\cos n\left(\pi - \frac{\pi}{n}\right) = -1 < 0, \qquad -\sigma_1 \cos(n-1)\left(\pi - \frac{\pi}{n}\right) < 0,$$

$$\sigma_2 \cos(n-2)\left(\pi - \frac{\pi}{n}\right) < 0, \dots,$$

$$(-1)^{n/2-1} \sigma_{n/2-1} \cos\left(\frac{n}{2} + 1\right)\left(\pi - \frac{\pi}{n}\right) < 0,$$

$$(-1)^{n/2} \sigma_{n/2} \cos \frac{n}{2}\left(\pi - \frac{\pi}{n}\right) = 0,$$

$$(-1)^{n/2+1} \sigma_{n/2+1} \cos\left(\frac{n}{2} - 1\right)\left(\pi - \frac{\pi}{n}\right) > 0, \dots,$$

$$-\sigma_{n-1} \cos\left(\pi - \frac{\pi}{n}\right) > 0, \qquad \sigma_n > 0.$$

By the well-known Descartes's rule of signs (see [7, p. 60]), $U(x)$ has a unique positive root (denoted by r_0). When r increases through r_0, $U(r)$

changes from positive to 0, then to negative. And $W(r) < 0$ for any $r \in (0, +\infty)$; therefore

$$I_0^\infty \frac{W(r)}{U(r)} = 1.$$

(ii) If n is odd, then by reasoning similar to (i), it is known that all coefficients of $W(r)$ are positive except the coefficient 0 of r^n. Hence for any $r \in (0, +\infty)$, $W(r) > 0$. Meantime, the coefficients of $r^n, r^{n-1}, \dots, r^{(n+1)/2}$ in $U(r)$ are positive; the coefficients of $r^{(n-1)/2}, \dots, r^2, r$ and the constant term are negative. By the Descartes's rule of signs, $U(x)$ has a unique positive root. When r increases through this root, $U(x)$ changes from negative to 0, then to positive. And $W(r) > 0$, for any $r \in (0, +\infty)$. Therefore

$$I_0^\infty \frac{W(r)}{U(r)} = 1.$$

By (i) and (ii), for any $n = 1, 2, \dots, I_0^\infty W(r)/U(r) = 1$. By Corollary 1, the number of u in S such that $|\arg u| < \pi - \pi/n$ is

$$K = I_0^\infty \frac{W(r)}{U(r)} + \frac{n}{\pi}\left(\pi - \frac{\pi}{n}\right) - \frac{1}{\pi} \arctan \tan n\left(\pi - \frac{\pi}{n}\right) = n.$$

Hence for any $i = 1, \dots, n$, $|\arg u_i| < \pi - \pi/n$.

(II) Since $u_i \neq 0$ $(i = 1, \dots, n)$, $\sigma_n > 0$. If there exists some j $(1 \leqslant j < n)$ such that $\sigma_j \neq 0$, then, when n is even, similarly, we obtain that for any $r \in (0, +\infty)$ we have $W(r) < 0$, and $U(x)$ has a unique positive root such that when r increases through it, $U(x)$ changes from positive to 0, then to negative; hence $I_0^\infty W(r)/U(r) = 1$. When n is odd, we get also $I_0^\infty W(r)/U(r) = 1$. Furthermore, by Corollary 1, for any $i = 1, \dots, n$,

$$|\arg u_i| < \pi - \frac{\pi}{n}.$$

If $\sigma_j = 0$ for $j = 1, \dots, n-1$, then $f(Z) = Z^n + (-1)^n \sigma_n$. Let $f(Z) = 0$; then $Z^n = (-1)^{n+1}\sigma_n$. When n is odd, the n roots of $f(Z)$ can be shown to be

$$u_k = \sqrt[n]{\sigma_n}\, e^{i2k\pi/n}, \qquad k = 0, \pm 1, \pm 2, \dots, \pm \frac{n-1}{2}.$$

Hence

$$|\arg u_k| = \frac{2|k|\pi}{n} \leqslant \frac{2\dfrac{n-1}{2}\pi}{n} = \pi - \frac{\pi}{n},$$

where $|\arg u_{\pm(n-1)/2}| = \pi - \pi/n$. Therefore some equalities hold in (1′). When n is even, the n roots of $f(Z)$ can be denoted as

$$u_k = \sqrt[n]{\sigma_n}\, e^{i(2k\pi+\pi)/n}, \qquad k = 0, \pm 1, \pm 2, \dots, \pm\left(\frac{n}{2} - 1\right), -\frac{n}{2}.$$

Hence $-n/2 \leqslant k \leqslant n/2 - 1$. This implies $-(\pi - \pi/n) \leqslant (\pi + 2k\pi)/n \leqslant \pi - \pi/n$. Then

$$|\arg u_k| = \left|\frac{\pi + 2k\pi}{n}\right| \leqslant \pi - \frac{\pi}{n}$$

and

$$|\arg u_{-n/2}| = \left|-\left(\pi - \frac{\pi}{n}\right)\right| = \pi - \frac{\pi}{n}, \qquad |\arg u_{n/2-1}| = \pi - \frac{\pi}{n}.$$

Also, some equalities hold in (1′). ■

3. ON THE CONJECTURES OF THE LOCALIZATION OF THE SPECTRA OF P- AND P_0-MATRICES

In this section, we shall discuss the conjectures of D. Hershkowitz and A. Berman [4]. First, we consider Conjecture 1.

Let $S = \{u_1, \dots, u_n\}$ be a P-set, that is, $\sigma_k = \sigma_k(S) > 0$ for $k = 1, \dots, n$. Conjecture 1 says: if

(i) $\pi(S) = 2$, then
(ii) $|\arg u_i| < \frac{5}{6}\pi$ for $i = 1, \dots, n$.

We shall give a negative answer to Conjecture 1 with a counterexample such that (i) holds, but (ii) does not hold.

Let $n = 8$. Assume that $\sigma_1 = 4^{-8}$, $\sigma_2 = 4^3$, $\sigma_3 = 4^{-4}$, $\sigma_4 = 4^{-5}$, $\sigma_5 = 4^{-2}$, $\sigma_6 = 4^{-5}$, $\sigma_7 = 4^{-2}$, $\sigma_8 = 2\times 4^8 - \delta$ $(0 < \delta < 1)$. Let $u_1, \dots, u_8$ be all the roots of $f(Z) = Z^8 - \sigma_1 Z^7 + \cdots - \sigma_7 Z + \sigma_8$. Then $S = \{u_1, \dots, u_8\}$ is a P-set.

Let w be a real number. Then $f(iw)=g(w)+ih(w)$, where

$$g(w)=w^8-\sigma_2 w^6+\sigma_4 w^4-\sigma_6 w^2+\sigma_8$$

and

$$h(w)=\sigma_1 w^7-\sigma_3 w^5+\sigma_5 w^3-\sigma_7 w.$$

For $\sigma_8=2\times 4^8-\delta$, $g(w)$ takes on different values when δ is given different values, and any two of the different $g(w)$ have no common real roots. $h(w)$ has at most seven real roots. Hence there exist at most seven different δ such that the $g(w)$ determined by these δ and $h(w)$ have common real roots. Suppose these δ are $\delta_1,\dots,\delta_7$ (if they exist). Therefore if $\delta=\delta_0\in(0,1)$ such that $\delta_0\neq\delta_i$ for $i=1,\dots,7$, then $g(w)$ and $h(w)$ have no common real roots. Hence $f(iw)\neq 0$ for any $w\in R$, and $f(Z)$ has no roots on the imaginary axis. Thus $f(Z)$ satisfies the condition of Theorem 1 for $\theta=\pi/2$.

In the following we always take $\delta=\delta_0$, i.e. $\sigma_8=2\times 4^8-\delta_0$. Since $\pi(S)$ is actually the number of roots u of $f(Z)$ such that $|\arg u|<\pi/2$, by Corollary 1 we have

$$\pi(S)=I_0^\infty\frac{h(w)}{g(w)}+\frac{8\times\pi/2}{\pi}-\frac{1}{\pi}\arctan\tan\left(8\times\frac{\pi}{2}\right)=I_0^\infty\frac{h(w)}{g(w)}+4. \quad (17)$$

Now we seek the value of $I_0^\infty h(w)/g(w)$. We have

$$g(w)=w^8-4^3w^6+4^{-5}w^4-4^{-5}w^2+2\times 4^8-\delta_0,$$

$$g'(w)=8w^7-6\times 4^3w^5+4\times 4^{-5}w^3-2\times 4^{-5}w,$$

$$g''(w)=8\times 7w^6-6\times 5\times 4^3w^4+4\times 3\times 4^{-5}w^2-2\times 4^{-5},$$

$$g^{(3)}(w)=8\times 7\times 6w^5-6\times 5\times 4\times 4^3w^3+24\times 4^{-5}w,$$

$$g^{(4)}(w)=8\times 7\times 6\times 5w^4-6\times 5\times 4\times 3\times 4^3w^2+24\times 4^{-5},$$

$$g^{(5)}(w)=8\times 7\times 6\times 5\times 4w^3-6\times 5\times 4\times 3\times 2\times 4^3w,$$

$$g^{(6)}(w)=8\times 7\times 6\times 5\times 4\times 3w^2-6!\times 4^3,$$

$$g^{(7)}(w)=8!\times w,$$

$$g^{(8)}(w)=8!.$$

Let $V_g(w)$ denote the number of variations in sign in the sequence

$$g(w), g'(w), \ldots, g^{(8)}(w).$$

We get $V_g(0) = 4$, $V_g(+\infty) = 0$, and $V_g(4) = 1$. Then $V_g(0) - V_g(4) = 3$, $V_g(4) - V_g(+\infty) = 1$. By the well-known Budan-Fourier theorem (see [7, p. 59]), in $(4, +\infty)$ $g(w)$ has just one root and in $(0,4)$ it has one or three roots.

We prove that in $(0,4)$ $g(w)$ has either just one root or just a root of multiplicity 3. Otherwise, suppose in $(0,4)$ $g(w)$ has either (i) three different roots, or (ii) one simple root and one root of multiplicity 2.

For (i), assume that w_1, w_2, w_3 are those three different roots such that $w_1 < w_2 < w_3$. By the classical Rolle's theorem, there exist $w' \in (w_1, w_2)$ such that $g'(w') = 0$ and $w'' \in (w_2, w_3)$ such that $g'(w'') = 0$.

For (ii), assume that $w^{(1)}$ is the simple root and $w^{(2)}$ is the root of multiplicity 2. Then there exists w' between $w^{(1)}$ and $w^{(2)}$ such that $g'(w') = 0$. And $g(w) = \mu(w)(w - w^{(2)})^2$, where $\mu(w)$ is a polynomial satisfying $\mu(w^{(2)}) \neq 0$. Thus $g'(w^{(2)}) = 0$.

In a word, in (i) and (ii), $g'(w)$ must have two different roots in $(0,4)$. But $g''(w) = w^4(56w^2 - 15\times 4^3) + (-15\times 4^3 w^4 + 12\times 4^{-5}w^2 - 2\times 4^{-5})$, where $56w^2 - 15\times 4^3 < 0$ for any $w \in (0,4)$, and $-15\times 4^3 w^4 + 12\times 4^{-5}w^2 - 2\times 4^{-5} = -15\times 4^3(w^2 - \frac{6}{15}4^{-8})^2 + \frac{36}{15}4^{-13} - 2\times 4^{-5} < 0$. Hence for any $w \in (0,4)$, $g''(w) < 0$, i.e., $g'(w)$ is a strictly monotone decreasing function in $(0,4)$. Therefore $g'(w)$ has at most one root in $(0,4)$. This contradicts (i) and (ii).

Thus, in $(0,4)$, $g(w)$ has just one root (a simple root or a root of multiplicity 3).

Now we discuss the localization of roots of $h(w)$ in $(0, +\infty)$.

By Descartes's rule of signs, $h(w) = 4^{-8}w^7 - 4^{-4}w^5 + 4^{-2}w^3 - 4^{-2}w$ has either three positive roots or one positive root. It will be verified that $h(w)$ has three different positive simple roots.

Obviously, $w = 4$ is a root of $h(w)$. And $h(2) > 0$, but $h(w) < 0$ if w is an enough small positive number; hence $h(w)$ has at least one root in $(0,2)$. Also, $h(4^{3/2}) < 0$ and $h(+\infty) = +\infty$; hence $h(w)$ has at least one root in $(4^{3/2}, +\infty)$. But $h(w)$ has at most three positive roots. Therefore, besides $w = 4$, $h(w)$ has a simple root in $(0,2)$ and one in $(4^{3/2}, +\infty)$. Then $h(w)$ has no roots in $(2,4)$ and $(4, 4^{3/2})$, and

$$h(w) > 0 \quad \text{if} \quad w \in (2,4), \tag{18}$$

$$h(w) < 0 \quad \text{if} \quad w \in (4, 4^{3/2}). \tag{19}$$

We have $g(2) > 0$, $g(4^{3/2}) > 0$, and $g(4) < 0$. It has been proved that $g(w)$ has one root in $(0,4)$ and one in $(4, +\infty)$ (simple or multiplicity 3). As a result, $g(w)$ has just one root (simple or multiplicity 3) in $(2,4)$ and one in $(4, 4^{3/2})$, and has no other positive roots.

Let w^* be the root of $g(w)$ in $(2,4)$. Then if w increases through w^*, $g(w)$ changes from positive to 0, then to negative. And by (18), $h(w^*) > 0$. Hence w^* is a jump of $h(w)/g(w)$ from $+\infty$ to $-\infty$.

Let w^{**} be the root in $(4, 4^{3/2})$ of $g(w)$. Then if w increases through w^{**}, $g(w)$ changes from negative to 0, then to positive. And by (19), $h(w^{**}) < 0$. Hence w^{**} is also a jump of $h(w)/g(w)$ from $+\infty$ to $-\infty$.

$h(w)/g(w)$ has no other jumps, since $g(w)$ has no positive roots except w^* and w^{**}. By the definition of the Cauchy index, we have

$$I_0^\infty \frac{h(w)}{g(w)} = -2.$$

By (17), $\pi(S) = I_0^\infty h(w)/g(w) + 4 = -2 + 4 = 2$; then $S = \{u_1, \dots, u_8\}$ satisfies condition (i) of Conjecture 1.

Now, we check condition (ii) for S. By Theorem 1, a necessary result of (ii) is (for $\theta = \frac{5}{6}\pi$)

$$8 = I_0^\infty \frac{W(w)}{U(w)} + \frac{8 \times \frac{5}{6}\pi}{\pi} - \frac{1}{\pi} \arctan\tan\left(8 \times \tfrac{5}{6}\pi\right),$$

or equivalently,

$$I_0^\infty \frac{W(w)}{U(w)} = 1, \tag{20}$$

where

$$\begin{aligned} W(w) = {} & w^8 \sin\left(8 \times \tfrac{5}{6}\pi\right) - \sigma_1 w^7 \sin\left(7 \times \tfrac{5}{6}\pi\right) \\ & + \sigma_2 w^6 \sin\left(6 \times \tfrac{5}{6}\pi\right) - \sigma_3 w^5 \sin\left(5 \times \tfrac{5}{6}\pi\right) \\ & + \sigma_4 w^4 \sin\left(4 \times \tfrac{5}{6}\pi\right) - \sigma_5 w^3 \sin\left(3 \times \tfrac{5}{6}\pi\right) \\ & + \sigma_6 w^2 \sin\left(2 \times \tfrac{5}{6}\pi\right) - \sigma_7 w \sin \tfrac{5}{6}\pi \end{aligned}$$

$$= \frac{\sqrt{3}}{2}w^8 + \frac{4^{-8}}{2}w^7 - \frac{4^{-4}}{2}w^5 - \frac{\sqrt{3}}{2}4^{-5}w^4$$

$$- 4^{-2}w^3 - \frac{\sqrt{3}}{2}4^{-5}w^2 - \frac{4^{-2}}{2}w,$$

$$U(w) = w^8\cos\left(8\times\tfrac{5}{6}\pi\right) - \sigma_1 w^7\cos\left(7\times\tfrac{5}{6}\pi\right)$$

$$+ \sigma_2 w^6\cos\left(6\times\tfrac{5}{6}\pi\right) - \sigma_3 w^5\cos\left(5\times\tfrac{5}{6}\pi\right)$$

$$+ \sigma_4 w^4\cos\left(4\times\tfrac{5}{6}\pi\right) - \sigma_5 w^3\cos\left(3\times\tfrac{5}{6}\pi\right) + \sigma_6 w^2\cos\left(2\times\tfrac{5}{6}\pi\right) - \sigma_7 w\cos\tfrac{5}{6}\pi + \sigma_8$$

$$= -\tfrac{1}{2}w^8 - \frac{\sqrt{3}}{2}4^{-8}w^7 - 4^3w^6 - \frac{\sqrt{3}}{2}4^{-4}w^5 - \tfrac{1}{2}4^{-5}w^4$$

$$+ \tfrac{1}{2}4^{-5}w^2 + \frac{\sqrt{3}}{2}4^{-2}w + 2\times4^8 - \delta_0.$$

By Descartes's rule of signs, $W(w)$ and $U(w)$ each have just one positive simple root.

Since $W(1)>0$ and $W(+\infty)=+\infty$, the unique positive root of $W(w)$ lies in $(0,1)$, and

$$W(w)>0 \qquad \text{if} \quad w\in(1,+\infty). \tag{21}$$

Since $U(1)>0$ and $U(+\infty)=-\infty$, the unique positive root of $U(w)$ lies in $(1,+\infty)$. Let w_0 be this root. If w increases in $(1,+\infty)$ through w_0, $U(w)$ changes from positive to 0, then to negative. By (21), $W(w_0)>0$. Hence w_0 is a jump of $W(w)/U(w)$ from $+\infty$ to $-\infty$. But w_0 is the unique jump of $W(w)/U(w)$, since $U(w)$ has no positive roots except w_0. Hence

$$I_0^\infty \frac{W(w)}{U(w)} = -1.$$

This contradicts (20).

Thus $S=\{u_1,\dots,u_8\}$ does not satisfy condition (ii) of Conjecture 1. It follows that Conjecture 1 does not hold.

In a similar way, we can construct a counterexample of Conjecture 2. Let $S=\{u_1,\dots,u_8\}$ be a P-set. Conjecture 2 says: if

(i) ${}_{\gamma}(S)=2$, then
(ii) $|\arg u_i|<\frac{5}{6}\pi$, $i=1,\dots,n$.

Let $n = 8$. Assuming that $\sigma_1 = 4^{-2}$, $\sigma_2 = 4^3$, $\sigma_3 = 4^{-4}$, $\sigma_4 = 4^{-4}$, $\sigma_5 = 4^{-2}$, $\sigma_6 = 4^{-4}$, $\sigma_7 = 4^4$, $\sigma_8 = 2\times 4^8 - \delta$ $(0 < \delta < 1)$. Then S is the set of all roots of $f(Z) = Z^8 - \sigma_1 Z^7 + \cdots - \sigma_7 Z + \sigma_8$ and is a P-set. As in the foregoing paragraphs, there exists $\delta_0 \in (0,1)$ such that $g(w)$ and $h(w)$ have no common root. In the following, we always take $\delta = \delta_0$. Then, for any $w \in (-\infty, +\infty)$, $f(iw) = g(w) + ih(w) \neq 0$. It follows that $\pi(S) + {}_{\surd}(S) = |S|$. Then $\pi(S) = 8 - {}_{\surd}(S)$. By Theorem 1,

$$\pi(S) = I_0^{\infty}\frac{h(w)}{g(w)} + \frac{8\times \pi/2}{\pi}\frac{1}{\pi}\arctan\tan\left(8\times\frac{\pi}{2}\right) = I_0^{\infty}\frac{h(w)}{g(w)} + 4.$$

Therefore

$$_{\surd}(S) = 4 - I_0^{\infty}\frac{h(w)}{g(w)}. \tag{22}$$

It can also be proved that in $(0,4)$ and in $(4, +\infty)$ $g(w)$ has just one root each (simple or multiplicity 3). And for any $w \in (0, +\infty)$,

$$h''(w) = 42\times 4^{-2}w\left(w^2 - \tfrac{10}{42}\times 4^{-2}\right)^2 + \left(6\times 4^{-2} - \tfrac{100}{42}\times 4^{-6}\right)w > 0.$$

Hence for the same reason as with $g(w)$ in $(0,4)$, $w = 4$ is the unique root in $(0, +\infty)$ of $h(w)$ (simple or multiplicity 3). Clearly,

$$\text{for any } w \in (0,4), \quad h(w) < 0; \qquad \text{for any } w \in (4, +\infty), \quad h(w) > 0.$$

And $g(0) > 0$, $g(4) < 0$, $g(+\infty) = +\infty$. Thus we see easily that the only two jumps of $h(w)/g(w)$ in $(0, +\infty)$ are both from $-\infty$ to $+\infty$. Hence $I_0^{\infty}h(w)/g(w) = 2$. By (22), ${}_{\surd}(S) = 4 - I_0^{\infty}h(w)/g(w) = 2$. This means that condition (i) of Conjecture 2 holds for $S = \{u_1, \dots, u_8\}$.

Now, $U(w)$ has an unique positive root in $(4^{3/4}, +\infty)$. So has $W(w)$ in $(0, 4^{3/4})$. And if $w \in (4^{3/4}, +\infty)$, then $W(w) > 0$.

We have $U(+\infty) = -\infty$, $U(4^{3/4}) > 0$. Thus $W(w)/U(w)$ has a unique positive jump from $+\infty$ to $-\infty$. Then $I_0^{\infty}W(w)/U(w) = -1$.

This contradicts the necessary result (20) of (ii). It means that $S = \{u_1, \dots, u_8\}$ does not satisfy condition (ii) of Conjecture 2. Therefore Conjecture 2 is false.

REMARK 1. Both $S = \{u_1, \dots, u_8\}$ in the counterexamples of Conjecture 1 and 2 are P_0-sets clearly. It is known that (ii) of Conjecture 1 and 2 is not satisfied by either S. But do the inequalities $|\arg u_i| \leqslant \frac{5}{6}\pi$, $i = 1,\dots,8$, hold or not?

Actually, if $|\arg u_i| \leqslant \frac{5}{6}\pi$, $i = 1,\dots,8$, then since (ii) is negative for the given S, we know that there exists at least one root $u_0 \in S$ of $f(Z)$ with $\arg u_0 = \pm\frac{5}{6}\pi$.

Let $u_0 = w_0 e^{\pm\frac{5}{6}\pi i}$, where $w_0 > 0$. Then $0 = f(u_0) = f(w_0 e^{\pm\frac{5}{6}\pi i}) = U(w_0) \pm iW(w_0)$ implies $U(w_0) = W(w_0) = 0$, that is, w_0 is a common positive root of $U(w)$ and $W(w)$.

In the counterexample of Conjecture 1, $U(w)$ has a unique positive root in $(1, +\infty)$ and $W(w)$ has a unique positive root in $(0,1)$. Hence $U(w)$ and $W(w)$ have no common positive roots. Therefore it is impossible for w_0 to be a common positive root of $U(w)$ and $W(w)$. This implies that the condition $|\arg u_0| \leqslant \frac{5}{6}\pi$, $i = 1,\dots,8$, is not satisfied by either S. Then both conjectures on P_0-sets corresponding to Conjectures 1 and 2 are false.

REMARK 2. A negative answer to Conjectures 2 and 4 (i.e. Conjectures 2 and 5 in [4]) was given by D. Hershkowitz and C. R. Johnson in [5]. But the method of constructing the counterexample of Conjecture 2 in [5] differs from that in this paper.

We wonder whether there exists a fixed $\theta \in (0, \pi)$ such that Conjecture 1 or 2 is true if θ replaces $\frac{5}{6}\pi$.

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REFERENCES

1 A. F. Beardon, *Complex Analysis*, Wiley-Interscience, New York, 1979.
2 F. R. Gantmacher, *The Theory of Matrices*, Vol. II, Chelsea, New York, 1959.
3 D. Hershkowitz, On the spectra of matrices having nonnegative sums of principal minors, *Linear Algebra Appl.* 55:81–86 (1983).
4 D. Hershkowitz and A. Berman, Localization of the spectra of P- and P_0-matrices, *Linear Algebra Appl.* 52/53:383–397 (1983).
5 D. Hershkowitz and C. R. Johnson, Spectra of matrices with P-matrix powers, *Linear Algebra Appl.* 80:159–171 (1986).
6 R. B. Kellogg, On complex eigenvalues of M and P matrices, *Numer. Math.* 19:170–175 (1972).
7 C. C. MacDuffee, *Theory of Equations*, Wiley, New York, 1953.

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